

FULLY FAITHFUL FOURIER-MUKAI FUNCTORS AND GENERIC VANISHING

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In memory of Alexandru T. Lascu

ABSTRACT. The aim of this mainly expository note is to point out that, given an Fourier-Mukai functor, the condition making it fully faithful is an instance of *generic vanishing*. We test this point of view on some fairly classical examples, including the strong simplicity criterion of Bondal and Orlov, the standard flip and the Mukai flop.

The aim of this mainly expository note is to point out that, given an Fourier-Mukai functor, the condition making it fully faithful is an instance of *generic vanishing*. We test this point of view on some fairly classical examples, including the strong simplicity criterion of Bondal and Orlov, the standard flip and the Mukai flop.

The notion of generic vanishing arose in work of Green and Lazarsfeld on irregular varieties ([GL1],[GL2]), where they showed that the sheaves of holomorphic differential forms, twisted with a generic topologically trivial line bundle, satisfy a cohomological vanishing of Kodaira-Nakano type. The natural environment for the notion of generic vanishing introduced by Green and Lazarsfeld is the Fourier-Mukai functor defined by the Poincaré line bundle. It makes sense to study the same kind of property for any FM functor ([PPo4],[Po]).

In this note we remark that a FM functor $\Phi_{\mathcal{E}}^{X \rightarrow Y} : D^b(X) \rightarrow D^b(Y)$ is fully faithful if and only if \mathcal{O}_Y , the structure sheaf of Y , satisfies Green-Lazarsfeld's generic vanishing condition (in the current terminology: is a geometric GV-object) with respect to the FM functor

$$\Phi_{\mathcal{E} \boxtimes_Y \mathcal{E}^\vee}^{Y \rightarrow X \times X} : D^b(Y) \rightarrow D^b(X \times X)$$

(see below for the notation), plus an additional condition which is usually easier to check. In essence, to be fully faithful is very close to be a generic vanishing condition. While this is certainly not a new result, but just a restatement of well known basic facts, it is the author's hope that this point of view can be an useful complement to the existing methods of investigating whether a given FM functor is fully faithful, in particular an equivalence. We test this by providing different proofs of some fairly classical full-faithfulness results.

Here is what the reader will find in this paper. The first section is background on fully faithful FM functors. The second section is background about generic vanishing conditions: they are usually stated in three equivalent ways, which we recall. In Section 3 we show that, in the context of full-faithfulness, the first equivalent condition essentially boils down to the strong simplicity criterion of Bondal and Orlov. Interestingly, the natural version of Bondal-Orlov's criterion in this context works under weaker hypotheses (Prop. 3.1). In Section 4 we use the second equivalent condition to give, or outline, alternative proofs about full-faithfulness of the natural FM functors associated

to standard flips and Mukai flops. Finally, in the last section we prove a full-faithfulness criterion (Prop. 5.1) corresponding the third equivalent way of expressing generic vanishing, and we illustrate it in the example of Poincaré kernels.

Notation. (a) All functors (as f^* , f_* , \otimes , ...) denote the functor on the derived category of coherent sheaves. For example \otimes means $\otimes^{\mathbf{L}}$, and the underived tensor product of coherent sheaves is denoted tor_0 . Moreover H^i means hypercohomology.

(b) Unless otherwise stated, all varieties are assumed smooth and projective over an algebraically closed ground field.

This paper is dedicated to the memory of Alexandru Lascu, my teacher and adviser back when I was an undergraduate at the University of Ferrara.

1. FULLY FAITHFUL FOURIER-MUKAI FUNCTORS

Let X and Y be smooth projective varieties and

$$\Phi_{\mathcal{E}}^{X \rightarrow Y} : D^b(X) \rightarrow D^b(Y)$$

a Fourier-Mukai functor of kernel $\mathcal{E} \in D^b(X \times Y)$. We denote

$$(1.1) \quad \mathcal{E}^\vee = R\mathcal{H}om(\mathcal{E}, \mathcal{O}_{X \times Y})$$

and p_X and p_Y the projections of $X \times Y$. The functor

$$\Phi_{\mathcal{E}^\vee \otimes p_Y^* \omega_Y[\dim Y]}^{Y \rightarrow X} : D^b(Y) \rightarrow D^b(X)$$

is the left adjoint of $\Phi_{\mathcal{E}}$. It follows that $\Phi_{\mathcal{E}}$ is fully faithful if and only the natural morphism of functors

$$(1.2) \quad \Phi_{\mathcal{E}^\vee \otimes p_Y^* \omega_Y[\dim Y]}^{Y \rightarrow X} \circ \Phi_{\mathcal{E}}^{X \rightarrow Y} \longrightarrow \text{id}_{D^b(X)}$$

is an isomorphism. The functor $\Phi_{\mathcal{E}^\vee \otimes p_Y^* \omega_Y[\dim Y]}^{Y \rightarrow X} \circ \Phi_{\mathcal{E}}^{X \rightarrow Y}$ is the FM functor of kernel

$$\Phi_{\mathcal{E} \boxtimes (\mathcal{E}^\vee \otimes p_Y^* \omega_Y)[\dim Y]}^{Y \times Y \rightarrow X \times X}(\mathcal{O}_{\Delta_Y})$$

(e.g. [Hu] Ex. 5.13(ii)). Therefore, since the unique kernel for $\text{id}_{D^b(X)}$ is \mathcal{O}_{Δ_X} , the functor $\Phi_{\mathcal{E}}^{X \rightarrow Y}$ is fully faithful if and only if

$$(1.3) \quad \Phi_{\mathcal{E} \boxtimes (\mathcal{E}^\vee \otimes p_Y^* \omega_Y)[\dim Y]}^{Y \times Y \rightarrow X \times X}(\mathcal{O}_{\Delta_Y}) = \mathcal{O}_{\Delta_X}[-\dim Y]$$

Given \mathcal{E} and \mathcal{F} objects of $D^b(X \times Y)$, let $\mathcal{E} \boxtimes_Y \mathcal{F}$ be the object of $D^b(X \times X \times Y)$ defined as

$$\mathcal{E} \boxtimes_Y \mathcal{F} = p_{13}^* \mathcal{E} \otimes p_{23}^* \mathcal{F}$$

where p_{13} and p_{23} are the two projections of $(X \times Y) \times_Y (X \times Y) = X \times X \times Y$. Since $\Phi_{\mathcal{E} \boxtimes_Y \mathcal{F}}^{Y \times Y \rightarrow X \times X}(\mathcal{O}_{\Delta_Y}) = \Phi_{\mathcal{E} \boxtimes_Y \mathcal{F}}^{Y \rightarrow X \times X}(\mathcal{O}_Y)$ the above condition (1.3) can be also written as follows

$$(1.4) \quad \Phi_{\mathcal{E} \boxtimes_Y (\mathcal{E}^\vee \otimes p_Y^* \omega_Y)}^{Y \rightarrow X \times X}(\mathcal{O}_Y) = \mathcal{O}_{\Delta_X}[-\dim Y]$$

or also

$$(1.5) \quad \Phi_{\mathcal{E} \boxtimes_Y \mathcal{E}^\vee}^{Y \rightarrow X \times X}(\omega_Y) = \mathcal{O}_{\Delta_X}[-\dim Y]$$

2. GENERIC VANISHING

Let Y and Z be smooth projective varieties and $\mathcal{P} \in D^b(Y \times Z)$. Let $z \in Z$ be a closed point and k_z its residue field, seen a coherent sheaf on Z supported at z . We have that $\Phi_{\mathcal{P}}^{Z \rightarrow Y}(k_z) = i_z^* \mathcal{P}$, where $i_z : Y \rightarrow Y \times Z$ is $i_z(y) = (y, z)$. We consider now the Fourier-Mukai functor in the opposite direction, $\Phi_{\mathcal{P}}^{Y \rightarrow Z} : D^b(Y) \rightarrow D^b(Z)$. Given $F \in D^b(Y)$, we define its cohomological support loci with respect to $\Phi_{\mathcal{P}}^{Y \rightarrow Z}$ as

$$V_{\mathcal{P}}^i(Y, F) = \{z \in Z \mid h^i(Y, F \otimes \Phi_{\mathcal{P}}^{Z \rightarrow Y}(k_z)) > 0\}$$

Example 2.1. (*Green-Lazarsfeld sets*) Let Y be an irregular variety, $Z = \text{Pic}^0 Y$ and \mathcal{P} a Poincaré line bundle on $Y \times \text{Pic}^0 Y$. Then $z \in \text{Pic}^0 Y$ corresponds to a line bundle P_z on Y , which is precisely $\Phi_{\mathcal{P}}^{\text{Pic}^0 Y \rightarrow Y}(k_z)$. Given a coherent sheaf F on Y , the cohomological support loci

$$V_{\mathcal{P}}^i(Y, F) = \{z \in \text{Pic}^0 Y \mid h^i(F \otimes P_z) > 0\}$$

were introduced and studied by Green and Lazarsfeld for the sheaves $F = \Omega_Y^j$ ([GL1], [GL2]) and subsequently studied for other relevant sheaves on abelian and/or irregular varieties, see e.g. [PPo1], [PPo2].

The following notion was introduced by Mihnea Popa in [Po], Def. 3.7.

Definition 2.2. (*Geometric GV-objects*) An object $F \in D^b(Y)$ is called a *geometric GV-object* with respect to a functor $\Phi_{\mathcal{P}}^{Y \rightarrow Z}$ if

- (i) $V_{\mathcal{P}}^i(Y, F) = \emptyset$ for $i < 0$ and
- (ii) $\text{codim}_Z V_{\mathcal{P}}^i(Y, F) \geq i$ for $i \geq 0$ ¹.

The following result is well known, see [PPo3], [PPo4] and especially [Po], Th. 3.8, Remark 3.10 and Cor. 4.3 and references therein. See also Remark 2.6 below.

Theorem 2.3. *In the above setting, the following are equivalent*

- (a) F is a geometric GV-object with respect to the functor $\Phi_{\mathcal{P}}^{Y \rightarrow Z}$;
- (b) $\Phi_{\mathcal{P}^\vee}^{Y \rightarrow Z}(F^\vee \otimes \omega_Y)$ is concentrated in cohomological degree $\dim Y$. That is:

$$\Phi_{\mathcal{P}^\vee}^{Y \rightarrow Z}(F^\vee \otimes \omega_Y) = R^{\dim Y} \Phi_{\mathcal{P}^\vee}^{Y \rightarrow Z}(F^\vee \otimes \omega_Y)[- \dim Y]$$

- (c) If A is a sufficiently high multiple of an ample line bundle on Z then

$$H^i(Y, \Phi_{\mathcal{P}^\vee}^{Z \rightarrow Y}(A) \otimes F^\vee \otimes \omega_Y) = 0 \quad \text{for all } i \neq \dim Y.^2$$

¹Note that in *loc cit* condition (i) is stated in a different way, namely $R^j \Phi_{\mathcal{E}^\vee \otimes p_Z^* \omega_Z}^{X \times Y}(F^\vee \otimes \omega_Y) = 0$ for $j > \dim Y$. This is equivalent to (i) by duality and base-change. Indeed by duality (i) is equivalent to the fact the loci $V_{\mathcal{P}^\vee \otimes p_Z^* \omega_Z}^j(Y, F^\vee \otimes \omega_Y)$ are empty for all $j > \dim Y$, which is in turn equivalent (by an easy application of base-change) to the vanishing of the sheaves $R^j \Phi_{\mathcal{E}^\vee \otimes p_Z^* \omega_Z}^{X \times Y}(F^\vee \otimes \omega_Y) = 0$ for all $j > \dim Y$.

²condition (c) is more usually expressed in the dual way, namely $H^i(Y, F \otimes \Phi_{\mathcal{P}[\dim Z]}^{Z \rightarrow Y}(A^\vee)) = 0$ for $i \neq 0$ (see e.e. [PPo4] Cor.3.11(b)). By duality and Serre vanishing it is easily seen that the two formulations are equivalent

According to a terminology/notation due to Mukai, condition (b) is sometimes referred to as the fact that $F^\vee \otimes \omega_Y$ satisfies the weak index theorem with index $i = \dim Y$. For short: $\text{WIT}(\dim Y)$. If this is the case the sheaf $R^{\dim Y} \Phi_{\mathcal{P}^\vee}^{Y \rightarrow Z}(F^\vee \otimes \omega_Y)$ is denoted $\widehat{\mathcal{F}^\vee \otimes \omega_Y}$. In this notation condition (b) is written as

$$(2.1) \quad \Phi_{\mathcal{P}^\vee}^{Y \rightarrow Z}(F^\vee \otimes \omega_Y) = \widehat{\mathcal{F}^\vee \otimes \omega_Y}[-\dim Y]$$

Remark 2.4. (*On the assumptions for Theorem 2.3*) Theorem 2.3 works under more general hypotheses: assuming that the kernel \mathcal{P} is a perfect complex, for the equivalence between (a) and (b) Z need not to be projective, and both Y and Z need not to be smooth varieties, but only Cohen-Macaulay schemes of finite type over any field (but, if Z is not Gorenstein, in condition (b) \mathcal{P}^\vee has to be replaced with $\mathcal{P}^\vee \otimes p_Z^* \omega_Z$ see [Po] Remark 3.10). The equivalence with (c) holds under the further assumption that Z is projective. We refer to [Po] and [PPo4].

Remark 2.5. (*Conditions (a) and (b) for (hyper)cohomology*) A simple-minded way to see the equivalence between (a) and (b) is as follows. Let Z be a point, denoted $\{pt\}$, and $\mathcal{P} = \mathcal{O}_{Y \times \{pt\}}$. The two conditions of the previous theorem are reduced to:

- (a₀) $H^i(Y, F) = 0$ for $i \neq 0$;
- (b₀) $H^i(Y, F^\vee \otimes \omega_Y) = 0$ for $i \neq \dim Y$.

They are equivalent by Serre duality, and the meaning of the equivalence between (a) and (b) is that, for arbitrary Fourier-Mukai functors, they admit distinct equivalent generalizations: the generalization of (a₀) is geometric-GV, namely the generic vanishing of a family of hypercohomology groups. The generalization of (b₀) is $\text{WIT}(\dim Y)$, that is the vanishing of the hyperdirect image sheaves $R^i \Phi_{\mathcal{P}^\vee}(F^\vee \otimes p_Y^* \omega_Y)$.

Remark 2.6. (*Perverse sheaves*) The geometric-GV and $\text{WIT}(\dim Y)$ conditions are better stated in terms of t-structures and perverse sheaves. We refer to [Po] and [PoS] §6-7 for this. Briefly, it follows from a result of Kashiwara ([K]) that a geometric GV-object with respect to $\Phi_{\mathcal{P}}^{Y \rightarrow Z}$ is an object F of $D^b(Y)$ such that $\Phi_{\mathcal{P}} F$ belongs to the heart of the dual t-structure on $D^b(Z)$. This is the equivalence between (a) and (b) in Theorem 2.3.

Remark 2.7. [*Condition (c) with ample sequences*] By duality,

$$(\Phi_{\mathcal{P}^\vee}^{Z \rightarrow Y}(A))^\vee \cong \Phi_{\mathcal{P}[\dim Z]}^{Z \rightarrow Y}(A^\vee \otimes \omega_Z)$$

Therefore condition (c) can be written as follows

$$\text{Hom}(\Phi_{\mathcal{P}[\dim Z]}^{Z \rightarrow Y}(A^{-1} \otimes \omega_Z), F^\vee \otimes \omega_Y[j]) = 0 \quad \text{for all } j \neq \dim Y$$

Note that, if A is ample and $k \gg$ then $L_k := A^{-k} \otimes \omega_Z$ is ample sequence in $\text{coh}(Z)$. Condition (c) of Theorem 2.3 can be stated more generally as follows: given an ample sequence $\{L_k\}$ in $\text{coh}(Z)$,

$$\text{Hom}(\Phi_{\mathcal{P}[\dim Z]}^{Z \rightarrow Y}(L_k), F^\vee \otimes \omega_Y[j]) = 0 \quad \text{for all } j \neq \dim Y \text{ and } k \ll$$

The equivalence with condition (b) of Theorem 2.3 is proved in the same way.

3. FULL FAITHFULNESS VIA CONDITION (a)

The relationship between full faithfulness and generic vanishing is in (1.5), which can be reformulated as follows:

$\Phi_{\mathcal{E}}$ is fully faithful if and only if ω_Y satisfies $\text{WIT}(\dim Y)$ with respect to $\Phi_{\mathcal{E} \boxtimes_Y \mathcal{E}^\vee}$ – that is condition (b) of Theorem 2.3 – and, in addition, its transform is the sheaf \mathcal{O}_{Δ_X} in cohomological degree $\dim Y$.

Experience shows that, usually, the more difficult part to be checked is the $\text{WIT}(\dim Y)$ condition, while the additional requirement is easier. With this in mind, Theorem 2.3 provides three distinct ways of checking full-faithfulness.

Condition (a) leads to the classical strong simplicity criterion of Bondal-Orlov (see [BO], [Br] and [Hu] §7.1). See also [HeLS] and [L] for generalizations). Actually one gets the result under weaker hypotheses. To this purpose, let us consider the loci

$$W_{\mathcal{E}}^i(Y) = \{(x, x') \in X \times X \mid \text{Hom}(\Phi_{\mathcal{E}}^{X \rightarrow Y}(k_x), \Phi_{\mathcal{E}}^{X \rightarrow Y}(k_{x'})) [i] \neq 0\}$$

Proposition 3.1. *Assume that $\text{char } k = 0$. Then $\Phi_{\mathcal{E}}^{X \rightarrow Y} : D^b(X) \rightarrow D^b(Y)$ is fully faithful if and only if the following conditions hold:*

- (a) $W_{\mathcal{E}}^i(Y)$ is empty for $i < 0$;
- (b) $\dim W_{\mathcal{E}}^i(Y) \leq 2 \dim X - i$ for all $i \geq 0$;
- (c) $\text{Hom}(\Phi_{\mathcal{E}}^{X \rightarrow Y}(k_x), \Phi_{\mathcal{E}}^{X \rightarrow Y}(k_{x'})) = k$ if $x = x'$ and 0 otherwise.

Proof. Since X is smooth

$$(3.1) \quad (\Phi_{\mathcal{E}}^{X \rightarrow Y}(k_x))^\vee \cong \Phi_{\mathcal{E}^\vee}^{X \rightarrow Y}(k_x)$$

for all $x \in X$. Therefore

$$\begin{aligned} \text{Hom}(\Phi_{\mathcal{E}}^{X \rightarrow Y}(k_x), \Phi_{\mathcal{E}}^{X \rightarrow Y}(k_{x'})) [i] &= \text{Ext}^i(\Phi_{\mathcal{E}}^{X \rightarrow Y}(k_x), \Phi_{\mathcal{E}}^{X \rightarrow Y}(k_{x'})) \cong \\ &\cong H^i(Y, \Phi_{\mathcal{E}^\vee}^{X \rightarrow Y}(k_x) \otimes \Phi_{\mathcal{E}}^{X \rightarrow Y}(k_{x'})) = H^i(Y, \Phi_{\mathcal{E} \boxtimes_Y \mathcal{E}^\vee}^{X \times X \rightarrow Y}(k_{(x', x)})) \end{aligned}$$

It follows that

$$W_{\mathcal{E}}^i(Y) = V_{\mathcal{E} \boxtimes_Y \mathcal{E}^\vee}^i(Y, \mathcal{O}_Y)$$

Therefore (a) and (b) mean exactly that \mathcal{O}_Y is a geometric GV-object with respect to $\Phi_{\mathcal{E} \boxtimes_Y \mathcal{E}^\vee}^{Y \rightarrow X \times X}$. By (a) \Leftrightarrow (b) of Theorem 2.3, this is equivalent to the fact that $\Phi_{\mathcal{E}^\vee \boxtimes_Y \mathcal{E}}^{Y \rightarrow X \times X}(\omega_Y)$ is a coherent sheaf on $X \times X$ in cohomological degree $\dim Y$ (note that $(\mathcal{E} \boxtimes_Y \mathcal{E}^\vee)^\vee \cong \mathcal{E}^\vee \boxtimes_Y \mathcal{E}$). According to notation (2.1), we denote $\widehat{\omega}_Y$ this coherent sheaf. Hypotheses (a) and (c) of the present Theorem imply, by cohomology and base change, that this sheaf is in fact a line bundle on a possibly non-reduced variety supported on the diagonal Δ_X^3 . But in fact, as the ground field is assumed to be algebraically closed of characteristic zero, actually

$$\widehat{\omega}_Y = \delta_{X*} L,$$

where $\delta_X : X \rightarrow X \times X$ is the diagonal embedding and L is a line bundle on X : this is proved exactly as in Bridgeland's account of Bondal-Orlov's theorem, using the Kodaira-Spencer map ([Br] Lemmas 5.2-3 or [Hu] Steps 3 and 5 or the proof of the main result in [L]), so we won't reproduce this argument here. This already proves that $\Phi_{\mathcal{E}}$ is fully faithful (and, a posteriori, $L = \mathcal{O}_X$). \square

³in fact one known that, since \mathcal{O}_Y is geometric-GV with respect to $\Phi_{\mathcal{P}}^{Y \rightarrow X \times X}$, the sheaf $\widehat{\omega}_Y$ has the "base-change property", namely, in the present case, the natural map $\text{tor}_0(\widehat{\omega}_Y, k_{(x, x')}) \rightarrow H^{\dim Y}(Y, \omega_Y \otimes \Phi_{\mathcal{E}^\vee \boxtimes_Y \mathcal{E}}^{X \times X \rightarrow Y}(k_{(x, x')}))$ is an isomorphism

Corollary 3.2. (*Bondal-Orlov*) $\Phi_{\mathcal{E}}$ is fully faithful if and only if

$$\mathrm{Hom}(\Phi_{\mathcal{E}}^{X \rightarrow Y}(k_x), \Phi_{\mathcal{E}}^{X \rightarrow Y}(k_{x'})[i]) = \begin{cases} k & \text{if } x = x' \text{ and } i = 0 \\ 0 & \text{if } x \neq x', \text{ or } i < 0, \text{ or } i > \dim X \end{cases}$$

Proof. The hypotheses can be restated as follows: $\mathrm{Hom}(\Phi_{\mathcal{E}}^{X \rightarrow Y}(k_x), \Phi_{\mathcal{E}}^{X \rightarrow Y}(k_{x'})) = k$ if $x = x'$ and

$$W_{\mathcal{E}}^i(Y) \begin{cases} = \emptyset & \text{for } i < 0 \\ \subseteq \Delta_X & \text{for } 0 \leq i \leq \dim X \\ = \emptyset & \text{for } i > \dim X. \end{cases}$$

Therefore Proposition 3.1 implies the Corollary. \square

Remark 3.3. (*On the assumptions for Prop. 3.1 and Corollary 3.2*) (i) As pointed out in [HeLS] Remark 1.25 and [L] the characteristic zero is necessary, unless one puts a supplementary hypothesis. (ii) Checking carefully more general assumptions for the validity of (1.2) and for the equivalence between (a) and (b) in Theorem 2.3, it follows that Prop. 3.1 and Corollary 3.2 work under more general hypotheses on X and Y : X needs to be smooth but not necessarily projective, while Y needs to be projective but it is allowed to be singular (Cohen-Macaulay). This is a result in [HeLS], see also [L] and references therein.

4. FULL FAITHFULNESS VIA CONDITION (B)

In this section we will consider some examples where, from the point of view of generic vanishing, the easiest way of proving/disproving full faithfulness is given by condition (1.5) at once, which corresponds to condition (b) of Theorem 2.3. This amounts to

$$(4.1) \quad R^i \Phi_{\mathcal{E} \boxtimes_Y \mathcal{E}^\vee}(\omega_Y) = \begin{cases} 0 & \text{for } i \neq \dim Y \\ \mathcal{O}_{\Delta_X} & \text{for } i = \dim Y \end{cases}$$

This is certainly a bit old fashioned (for example, it is the way Mukai originally showed in [M] that the Poincaré kernel provides a derived equivalence between dual abelian varieties) but however the proofs below are easy, self-contained and conceptually clear, and might provide a complementary insight on some aspects of Kawamata's conjecture $K\text{-equivalence} \Rightarrow D\text{-equivalence}$.

Example 4.1. (*Standard flip*) We consider a standard flip

$$(4.2) \quad \begin{array}{ccccc} & & E = \mathbb{P}^l \times \mathbb{P}^k & & \\ & \swarrow \pi_X & \downarrow & \searrow \pi_Y & \\ & & Z & & \\ & \swarrow p & & \searrow q & \\ \mathbb{P}^l & \hookrightarrow X & & Y & \hookleftarrow \mathbb{P}^k \end{array}$$

where $\mathcal{N}_{\mathbb{P}^l/X} = \mathcal{O}(-1)^{k+1}$ and $\mathcal{N}_{\mathbb{P}^k/Y} = \mathcal{O}(-1)^{l+1}$, so that the dimension of the varieties X , Y and Z is $d = k + l + 1$. The morphism π_X (resp. π_Y) is the blow up of \mathbb{P}^l (resp. \mathbb{P}^k). Note that the

functor $\Phi_{\mathcal{O}_Z}^{X \rightarrow Y}$ coincides with $q_* \circ p^*$. The result, again due to Bondal and Orlov [BO], is that: $\Phi_{\mathcal{O}_Z}^{X \rightarrow Y} : D^b(X) \rightarrow D^b(Y)$ is fully faithful if and only if $k \leq l$.

Let us prove this statement by verifying condition (4.1). We have that

$$\mathcal{O}_Z^\vee = \omega_Z \otimes \omega_{X \times Y}^{-1}[-d]$$

It follows that in this case condition (4.1) takes the form

$$(4.3) \quad R^i \Phi_{\mathcal{O}_Z \boxtimes_Y \mathcal{O}_Z}(\mathcal{O}_Y) = \begin{cases} 0 & \text{for } i \neq 0 \\ \delta_*(\omega_X) & \text{for } i = 0 \end{cases}$$

Enumerating the four factors of the product $X \times X \times Y \times Y$, we denote Z_{ij} the subvariety Z of $X_i \times Y_j$. By definition $\mathcal{O}_Z \boxtimes_Y \mathcal{O}_Z$ is the following derived tensor product in $D(X \times X \times Y \times Y)$:

$$\mathcal{O}_Z \boxtimes_Y \mathcal{O}_Z = (\mathcal{O}_{Z_{13}} \boxtimes \mathcal{O}_{Z_{24}}) \otimes (\mathcal{O}_{X \times X \times \Delta_{34} Y})$$

The intersection in $X \times X \times Y \times Y$ of the two smooth and irreducible subvarieties $Z_{13} \times Z_{24}$ and $X \times X \times \Delta_{34} Y$ is the fibred product $Z \times_Y Z$. It has two irreducible components:

$$(4.4) \quad (Z_{13} \times Z_{24}) \cap (X \times X \times \Delta_{34} Y) = (\Delta_{13,24} Z) \cup (E \times_{\mathbb{P}^k} E) = (\Delta_{12,34} Z) \cup (\mathbb{P}^l \times \mathbb{P}^l \times \Delta_{34} \mathbb{P}^k)$$

where $\Delta_{12,34} : (X \times Y) \hookrightarrow X \times X \times Y \times Y$ is the diagonal embedding $(x, y) \mapsto (x, x, y, y)$. Sometimes we will simply write the right hand side as

$$Z \cup \mathbb{P}^l \times \mathbb{P}^l \times \mathbb{P}^k$$

The two components are smooth and their intersection is $E = \Delta_{12,34} E = \Delta_{12} \mathbb{P}^l \times \Delta_{34} \mathbb{P}^k$.

The first component has the right codimension, namely $3d = 2d + d$, while the second component has codimension $3d - (l - 1)$ (hence it has the right codimension only for $l = 1$). Since (4.4) is the intersection of two smooth, hence locally complete intersection subvarieties of a smooth ambient variety, it follows that the higher $\text{tor}_i^{X \times X \times Y \times Y}(\mathcal{O}_{Z_{13} \times Z_{24}}, \mathcal{O}_{X \times X \times \Delta_{34} Y})$ are non-zero only if $l > 1$, and they are supported on $\mathbb{P}^l \times \mathbb{P}^l \times \mathbb{P}^k$. Moreover tor_1 is locally free of rank $l - 1$ and $\text{tor}_i = \wedge^i \text{tor}_1$ for $i \geq 1$. More precisely, we have the following

Claim 4.2. *for $i > 0$ $\text{tor}_i^{X \times X \times Y \times Y}(\mathcal{O}_{Z_{13} \times Z_{24}}, \mathcal{O}_{X \times X \times \Delta_{34} Y}) = \wedge^i(\mathcal{O}_{\mathbb{P}^l \times \mathbb{P}^l \times \mathbb{P}^k}(0, 0, 1)^{\oplus l-1})$*

Proof. We first compute

$$\begin{aligned} \text{tor}_i^{X \times X \times Y \times Y}(\mathcal{O}_{E_{13} \times E_{24}}, \mathcal{O}_{X \times X \times \Delta_{34} Y}) &= \text{tor}_i^{X \times X \times Y \times Y}(\mathcal{O}_{\mathbb{P}^l \times \mathbb{P}^l \times \mathbb{P}^k \times \mathbb{P}^k}, \mathcal{O}_{X \times X \times \Delta_{34} Y}) = \\ &= p_{34}^* \mathcal{N}_{\mathbb{P}^k/Y}^\vee = p_{34}^* \wedge^i(\mathcal{O}(1)^{\oplus l+1}) = \wedge^i(\mathcal{O}_{\mathbb{P}^l \times \mathbb{P}^l \times \mathbb{P}^k}(0, 0, 1)^{\oplus l+1}) \end{aligned}$$

The third equality follows from the general isomorphism $(\mathcal{F} \boxtimes \mathcal{G}) \otimes_{Y \times Y} \mathcal{O}_{\Delta_Y} = \mathcal{F} \otimes_Y \mathcal{G}$ (where for $F, G \in D(Y)$). Therefore $\text{tor}_i^{Y \times Y}(\mathcal{F} \boxtimes \mathcal{G}, \mathcal{O}_{\Delta_Y}) = \text{tor}_i^Y(F, G)$. In our case

$$\text{tor}_i^{Y \times Y}(\mathcal{O}_{\mathbb{P}^k} \boxtimes \mathcal{O}_{\mathbb{P}^k}, \mathcal{O}_{\Delta_Y}) \cong \text{tor}_i^Y(\mathcal{O}_{\mathbb{P}^k}, \mathcal{O}_{\mathbb{P}^k}) = \wedge^i \mathcal{N}_{\mathbb{P}^k/Y}^\vee.$$

Next we compute the difference between $\text{tor}_i^{X \times X \times Y \times Y}(\mathcal{O}_{Z_{13} \times Z_{24}}, \mathcal{O}_{X \times X \times \Delta_{34} Y})$ and $\text{tor}_i^{X \times X \times Y \times Y}(\mathcal{O}_{E_{13} \times E_{24}}, \mathcal{O}_{X \times X \times \Delta_{34} Y})$. This is achieved by tensoring with $\mathcal{O}_{X \times X \times \Delta_{34} Y}$ the two exact sequences

$$0 \rightarrow \mathcal{O}_{Z_{13}} \boxtimes \mathcal{O}_{Z_{24}}(-E_{24}) \rightarrow \mathcal{O}_{Z_{13} \times Z_{24}} \rightarrow \mathcal{O}_{Z_{13} \times E_{24}} \rightarrow 0$$

$$0 \rightarrow \mathcal{O}_{Z_{13}}(-E_{13}) \boxtimes \mathcal{O}_{E_{24}} \rightarrow \mathcal{O}_{Z_{13} \times E_{24}} \rightarrow \mathcal{O}_{E_{13} \times E_{24}} \rightarrow 0$$

The assertion follows after a little calculation with the first exact sequence. \square

Concerning the underived tensor product, since

$$\text{tor}_0(\mathcal{O}_{Z_{13}} \boxtimes \mathcal{O}_{Z_{24}}, \mathcal{O}_{X \times X \times \Delta_{34} Y}) = \mathcal{O}_{(\Delta_{12,34} Z) \cup (\mathbb{P}^l \times \mathbb{P}^l \times \Delta_{34} \mathbb{P}^k)}$$

we have the "Mayer-Vietoris" exact sequence

$$0 \rightarrow \text{tor}_0(\mathcal{O}_{Z_{13}} \boxtimes \mathcal{O}_{Z_{24}}, \mathcal{O}_{X \times X \times \Delta_{34} Y}) \rightarrow \mathcal{O}_{\Delta_{13,24} Z} \oplus (\mathcal{O}_{\mathbb{P}^l \times \mathbb{P}^l \times \Delta_{34} \mathbb{P}^k}) \rightarrow \mathcal{O}_{\Delta_{12} \mathbb{P}^l \times \Delta_{34} \mathbb{P}^k} \rightarrow 0$$

Since

$$\omega_Z|_E = \mathcal{O}(-l, -k)$$

from the Claim we get that, for $i > 0$

$$(4.5) \quad \text{tor}_i^{X \times X \times Y \times Y}(\mathcal{O}_{Z_{13}} \boxtimes \omega_{Z_{24}}, \mathcal{O}_{X \times X \times \Delta_{34} Y}) = \mathcal{O}_{\mathbb{P}^l \times \mathbb{P}^l \times \mathbb{P}^k}(0, -l, -k + i)^{\oplus \binom{l-1}{i}}$$

(in particular, it vanishes for $i > l - 1$). For $i = 0$ we have the exact sequence

$$(4.6) \quad 0 \rightarrow \text{tor}_0(\mathcal{O}_{Z_{13}} \boxtimes \omega_{Z_{24}}, \mathcal{O}_{X \times X \times \Delta_{34} Y}) \rightarrow \omega_{\Delta_{13,24} Z} \oplus \mathcal{O}_{\mathbb{P}^l \times \mathbb{P}^l \times \Delta_{34} \mathbb{P}^k}(0, -l, -k) \rightarrow \mathcal{O}_{\Delta_{12} \mathbb{P}^l \times \Delta_{34} \mathbb{P}^k}(-l, -k) \rightarrow 0$$

Applying $p_{X \times X*}$, i.e. p_{12*} , to (4.6) it follows easily that in any case

(4.7)

$$R^j p_{12*}(\text{tor}_0^{X \times X \times Y \times Y}(\mathcal{O}_{Z_{13}} \boxtimes \omega_{Z_{24}}, \mathcal{O}_{X \times X \times \Delta_{34} Y})) = R^j p_{12*}(\omega_{\Delta_{13,24} Z}) = \begin{cases} \omega_{\Delta_{12} X} & \text{for } j = 0 \text{ and } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

because p_{12} restricted to $\Delta_{13,24} Z$ is simply the birational morphism $p : Z \rightarrow X$. Hence the above tor_0 does not cause any obstruction to the validity of (4.3). On the other hand, applying p_{12*} to (4.5) one sees that the vanishing

$$(4.8) \quad R^j p_{12*}(\text{tor}_i^{X \times X \times Y \times Y}(\mathcal{O}_{Z_{13}} \boxtimes \omega_{Z_{24}}, \mathcal{O}_{X \times X \times \Delta_{34} Y})) = 0 \quad \text{for all } i > 0 \text{ and all } j$$

holds if and only if $k \geq l$. Via an easy spectral sequence, (4.7) and (4.8) prove that (4.3), i.e. full-faithfulness of $\Phi_{\mathcal{O}_Z}^{X \rightarrow Y} : D^b(X) \rightarrow D^b(Y)$, holds if $k \geq l$. In a similar way it follows also that the full-faithfulness does not hold for $k < l$.

Example 4.3. (*Mukai flop*) We follow the notation of [Hu], §11.4. We have the diagram

$$(4.9) \quad \begin{array}{ccccc} & & E \subset \mathbb{P} \times \mathbb{P}^\vee & & \\ & \swarrow \pi_X & \downarrow & \searrow \pi_Y & \\ & & Z & & \\ & \swarrow p & & \searrow q & \\ \mathbb{P} & \xrightarrow{\quad} & X & & Y \xleftarrow{\quad} \mathbb{P}^\vee \end{array}$$

and $\mathcal{N}_{\mathbb{P}|X} = \Omega_{\mathbb{P}}$, $\mathcal{N}_{\mathbb{P}^\vee|Y} = \Omega_{\mathbb{P}^\vee}$. Here $\dim X = 2n$ and $\mathbb{P} = \mathbb{P}^n$. The maps p (resp. q) is the blow-up of \mathbb{P} (resp. \mathbb{P}^\vee) and $E = \mathbb{P}(\Omega_{\mathbb{P}}) \subset \mathbb{P} \times \mathbb{P}^\vee$ is the incidence correspondence point-hyperplane. It is well known, by a result of Kawamata and Namikawa ([Kal],[N1]) that: *the functor $q_* \circ p^* = \Phi_{\mathcal{O}_Z} : D^b(X) \rightarrow D^b(Y)$ is not fully faithful.*

Let us check this within the method of the previous example. Exactly as above, the condition for full-faithfulness is (4.3), and one has to compute

$$(4.10) \quad \text{tor}_i^{X \times X \times Y \times Y}(\mathcal{O}_{Z_{13}} \boxtimes \omega_{Z_{24}}, \mathcal{O}_{X \times X \times \Delta_{34} Y})$$

Again the intersection in $X \times X \times Y \times Y$ of $Z_{13} \times Z_{24}$ and $X \times X \times \Delta_{Y_{34}}$ is the fibered product $Z \times_Y Z$, which has the two irreducible components:

$$(Z_{13} \times Z_{24}) \cap (X \times X \times \Delta_{Y_{34}}) = Z \times_Y Z = \Delta_{13,24} Z \cup (E_{13} \times_{\mathbb{P}^\vee} E_{24})$$

One can compute all tor sheaves (4.10) as in Claim 4.2 and the result is similar. It happens that higher tor 's (i.e. the sheaves (4.10) for $i > 0$) don't affect condition (4.3), namely

$$(4.11) \quad R^j p_{12*}(\text{tor}_i^{X \times X \times Y \times Y}(\mathcal{O}_{Z_{13}} \boxtimes \omega_{Z_{24}}, \mathcal{O}_{X \times X \times \Delta_{34} Y})) = 0 \quad \text{for all } i > 0 \text{ and all } j$$

We leave this to the reader.

The reason why (4.3) is not satisfied is in the underived tensor product

$$\text{tor}_0^{X \times X \times Y \times Y}(\mathcal{O}_{Z_{13}} \boxtimes \omega_{Z_{24}}, \mathcal{O}_{X \times X \times \Delta_{34} Y})$$

As in the previous example this sits in the exact sequence

$$(4.12) \quad 0 \rightarrow \text{tor}_0(\mathcal{O}_{Z_{13}} \boxtimes \omega_{Z_{24}}, \mathcal{O}_{X \times X \times \Delta_{34} Y}) \rightarrow \omega_{\Delta_{13,24} Z} \oplus ((p_{24}^* \omega_Z)|_{E_{13} \times_{\Delta_{24} \mathbb{P}^\vee} E_{34}}) \rightarrow (\omega_{\Delta_{13,24} Z})|_{\Delta_{13,24} E} \rightarrow 0$$

We apply $p_{X \times X*}$, i.e. p_{12*} to the above exact sequence. Since $(\omega_X)|_{\mathbb{P}}$ is trivial, we have that

$$(4.13) \quad (\omega_Z)|_E = \omega_E(-E) = \mathcal{O}_{\mathbb{P} \times \mathbb{P}^\vee}(-n, -n)|_E \otimes \mathcal{O}_{\mathbb{P} \times \mathbb{P}^\vee}(1, 1)|_E = \mathcal{O}_{\mathbb{P} \times \mathbb{P}^\vee}(-(n-1), -(n-1))|_E$$

It follows that, for all i , $R^i p_{12*}$ applied to the sheaf on the right of the exact sequence (4.12) is zero for all i . Therefore, to compute the higher direct images $R^i p_{12*}$ of the tor_0 on the left, it is enough to compute $R^i p_{12*}$ of the sheaf in the middle. This has two summands. Concerning the first one, as in the previous example there is nothing contradicting (4.3), since

$$(4.14) \quad R^i p_{12*}(\omega_{\Delta_{13,24} Z}) = \begin{cases} \omega_{\Delta_{12} X} & \text{for } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

Concerning the second summand, note that the fiber of the projection

$$(4.15) \quad p_{12} : E_{13} \times_{\Delta_{34} \mathbb{P}^\vee} E_{24} \rightarrow \mathbb{P} \times \mathbb{P} \subset X \times X$$

over a pair $(x, x') \in \mathbb{P} \times \mathbb{P}$, with $x \neq x'$, is the intersection of the two hyperplanes of \mathbb{P}^\vee corresponding to x and x' , that is a $\mathbb{P}^{n-2} \subset \mathbb{P}^\vee$. Now (4.13) tells that $p_{24}^* \omega_Z$, restricted to a general fiber of (4.15) is $\mathcal{O}_{\mathbb{P}^{n-2}}(-(n-1))$. Therefore $R^i p_{12*}$ applied to the second summand of the middle part of sequence (4.12) is zero for $i < n-2$ and *non-zero and supported on* $\mathbb{P} \times \mathbb{P}$ for $i = n-2$. By an easy spectral sequence this, together with (4.11) and (4.14), yields that $R^{n-2} \Phi_{\mathcal{O}_Z \boxtimes_Y \omega_Z}(\mathcal{O}_{\Delta_Y})$ is non-zero. Therefore (4.3) is not verified and $\Phi_{\mathcal{O}_Z} : D^b(X) \rightarrow D^b(Y)$ is not fully faithful.

With a similar, but more complicated, calculation one can prove directly the result of Kawamata and Namikawa ([Ka1],[N1], see also [Hu]) that $\Phi_{\mathcal{O}_{\tilde{Z}}} : \mathbf{D}^b(X) \rightarrow \mathbf{D}^b(Y)$ is fully faithful, where $\tilde{Z} = Z \cup (\mathbb{P} \times \mathbb{P}^\vee)$. As a disclaimer, we should point out that this method, applied to the stratified Atiyah flop and Mukai flop ([C],[Ka2], [Ma], [N2]) becomes much more complicated.

5. FULL-FAITHFULNESS VIA CONDITION (C)

In this section we use condition (c) of Theorem 2.3 to provide another way to check full-faithfulness. So far condition (c) has proven to be extremely useful for detecting generic vanishing when the kernel is a Poincaré line bundle. In fact, if X is an abelian variety, \mathcal{P} a Poincaré line bundle on $X \times \widehat{X}$ (where \widehat{X} is the dual abelian variety) and A is an ample line bundle on \widehat{X} , the object $\Phi_{\mathcal{P}^\vee}^{\widehat{X} \rightarrow X}(A)$, has a peculiar description, which can be seen as an effect of the "abelianity" of the context: $\Phi_{\mathcal{P}^\vee}^{\widehat{X} \rightarrow X}(A)$ is a locally free sheaf which is, up to pullback via the isogeny $\varphi_A : \widehat{X} \rightarrow X$ associated to A , sum of copies of the line bundle A^\vee (see e.g. [M], Prop. 3.11(1)). Therefore, in the case of Poincaré kernel on dual abelian varieties, condition (c) is a very effective way of reducing the GV condition to vanishing theorems. This idea, due to Hacon ([H]), is extremely fruitful in the study of the geometry of irregular varieties. It is an interesting problem to find an adequate description of the objects $\Phi_{\mathcal{P}^\vee}^{Z \rightarrow Y}(A)$ in other cases.

In the present context, condition (c) of Theorem 2.3 leads to the full-faithfulness criterion below. Due to the above reason, at present its range of applicability is confined to abelian or irregular varieties.

Proposition 5.1. *Let A be a sufficiently high power of an ample line bundle on $X \times X$. Then $\Phi_{\mathcal{E}} : D^b(X) \rightarrow D^b(Y)$ is fully faithful if and only if*

$$(5.1) \quad h^i(Y, \Phi_{\mathcal{E} \boxtimes_Y \mathcal{E}^\vee}^{X \times X \rightarrow Y}(A) \otimes \omega_Y) = \begin{cases} 0 & \text{for } i \neq \dim Y \\ h^0(A \otimes \mathcal{O}_{\Delta_X}) & \text{for } i = \dim Y \end{cases}$$

Proof. By the equivalence between (b) and (c) of Theorem 2.3 the first line above means that $\Phi_{\mathcal{E} \boxtimes_Y \mathcal{E}^\vee}(\omega_Y)$ is a sheaf in cohomological degree $\dim Y$, denoted $\widehat{\omega}_Y[-\dim Y]$ (according to notation (2.1)). Therefore the adjunction morphism (1.2) is, up to a shift

$$\Phi_{\widehat{\omega}_Y}^{X \rightarrow X} \rightarrow \Phi_{\Delta_X}^{X \rightarrow X}$$

This induces a morphism of $\mathcal{O}_{X \times X}$ -modules

$$(5.2) \quad \widehat{\omega}_Y \rightarrow \mathcal{O}_{\Delta_X}$$

which is surjective since, for all $x \in X$, the adjunction morphism $\Phi_{\widehat{\omega}_Y}^{X \rightarrow X}(k_x) \rightarrow k_x$ is non-zero, hence surjective.

We stop for a moment, to recall from [PPo4], Lemma 2.1 the functorial isomorphism, for all i and for all objects G (resp. A) of $D^b(Y)$ (resp. $D^b(Z)$)

$$(5.3) \quad H^i(Y, G \otimes \Phi_{\mathcal{E} \boxtimes_Y \mathcal{E}^\vee}^{X \times X \rightarrow Y}(A)) \cong H^i(X \times X, \Phi_{\mathcal{E} \boxtimes_Y \mathcal{E}^\vee}^{Y \rightarrow X \times X}(G) \otimes A)^4$$

Note that, via duality, (5.3) is a restatement of the description of the adjoints of Fourier-Mukai functors, but it is more simply proved by the fact that

$$R\Gamma(Y, G \otimes \Phi_{\mathcal{E} \boxtimes_Y \mathcal{E}^\vee}^{X \times X \rightarrow Y}(A)) \cong R\Gamma(Y \times X \times X, p_Y^* G \otimes (\mathcal{E} \boxtimes_Y \mathcal{E}^\vee) \otimes p_{X \times X}^*(A)) \cong R\Gamma(X \times X, \Phi_{\mathcal{E} \boxtimes_Y \mathcal{E}^\vee}^{Y \rightarrow X \times X}(G) \otimes A)$$

by Leray isomorphism and projection formula.

⁴this is also the key ingredient in the proof of the equivalence between (b) and (c) of Theorem 2.3

Going back to our proposition, given a line bundle A which is a sufficiently high power of an ample line bundle, from (5.3) and the second line of (5.1) we get

$$h^0(Y, \Phi_{\mathcal{E} \boxtimes_Y \mathcal{E}^\vee}^{X \times X \rightarrow Y}(A) \otimes \omega_Y) = h^0(X \times X, \widehat{\omega}_Y \otimes A) = h^0(X \times X, \mathcal{O}_{\Delta_X} \otimes A)$$

Therefore, since the morphism (5.2) is surjective, Serre's vanishing applied to its kernel yields that (5.2) is an isomorphism. Hence (1.5) is verified. This proves that (5.1) implies full-faithfulness of $\Phi_{\mathcal{E}}^{X \rightarrow Y}$. The other implication follows immediately from (5.3) and Serre's vanishing. \square

Example 5.2. (*Mukai's theorem on the Poincaré kernel*) To illustrate Prop. 5.1 let us take X an abelian variety, $Y = \widehat{X}$ and as \mathcal{P} a Poincaré line bundle. We will show that $\Phi_{\mathcal{P}}^{X \rightarrow \widehat{X}}$ is fully faithful. From this it follows that it is in fact an equivalence. Moreover, since $\mathcal{P}^\vee = (-id, id)^* = (id, -id)^* \mathcal{P}$, this proves also that

$$\Phi_{\mathcal{P}}^{X \rightarrow \widehat{X}} \circ \Phi_{\mathcal{P}}^{\widehat{X} \rightarrow X} = (-id)^*[-\dim X]$$

i.e. the theorem of Mukai in [M]. As in [Hu] Prop. 9.19, in characteristic zero one has a much easier proof, using Bondal-Orlov's strong simplicity criterion (see §3). The present proof works in any characteristic, as well as Mukai's original proof.

Let $L = \mathcal{O}_X(n\Theta)$ be a sufficiently high power ample line bundle on X . We take $A = L \boxtimes L$. By Proposition 5.1 it is sufficient to prove that

$$(5.4) \quad h^i(\widehat{X}, \Phi_{\mathcal{P} \boxtimes_{\widehat{X}} \mathcal{P}^\vee}^{X \times X \rightarrow \widehat{X}}(L \boxtimes L)) = \begin{cases} 0 & \text{if } i \neq \dim X \\ h^0(X, L^2) & \text{if } i = \dim X \end{cases}$$

Let

$$\varphi_L : X \rightarrow \widehat{X}$$

the isogeny associated to L . We have that

$$(id, \varphi_L)^* \mathcal{P} = (p_1 + p_2)^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}$$

where $p_1 + p_2 : X \times X \rightarrow X$ is the group law. Therefore, letting $\mathcal{F} = (p_1 + p_2)^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}$, by flat base change we have that

$$(5.5) \quad \varphi_L^*(\Phi_{\mathcal{P} \boxtimes_{\widehat{X}} \mathcal{P}^\vee}^{X \times X \rightarrow \widehat{X}}(L \boxtimes L)) = \Phi_{\mathcal{F} \boxtimes_X \mathcal{F}^\vee}^{X \times X \rightarrow X}(L \boxtimes L)$$

where $U \boxtimes_X V$ means $p_{12}^* U \otimes p_{23}^* V$, where p_{12} and p_{23} are the two projections of $(X \times X) \times_X (X \times X) = X \times X \times X$ (the fibred product is with respect to the second projection of the first factor and the first projection of the second factor). Therefore we must compute the right-hand side of (5.5). We have that

$$\mathcal{F} \boxtimes_X \mathcal{F}^\vee = (p_1 + p_2)^* L \otimes (p_2 + p_3)^* L^{-1} \otimes p_1^* L \otimes p_3^* L^{-1}$$

Therefore

$$(5.6) \quad \Phi_{\mathcal{F} \boxtimes_X \mathcal{F}^\vee}^{X \times X \rightarrow X}(L \boxtimes L) = p_{2*}((p_1 + p_2)^* L \otimes (p_2 + p_3)^* L^{-1} \otimes p_1^* L^2)$$

and by Serre vanishing (actually this is not needed in the case) the outcome is a sheaf:

$$p_{2*}((p_1 + p_2)^* L \otimes (p_2 + p_3)^* L^{-1} \otimes p_1^* L^2) = R^0 p_{2*}((p_1 + p_2)^* L \otimes (p_2 + p_3)^* L^{-1} \otimes p_1^* L^2)[0]$$

Via the automorphism $(p_1, p_1 + p_2, p_2 + p_3) : X \times X \times X \rightarrow X \times X \times X$, i.e. $(x, y, z) \mapsto (x, x + y, y + z)$, the line bundle $(p_1 + p_2)^* L \otimes (p_2 + p_3)^* L^{-1} \otimes p_1^* L^2$ is identified to $L^2 \boxtimes L \boxtimes L^{-1}$. Hence

$$h^i(X, \Phi_{\mathcal{F} \boxtimes_{\hat{X}} \mathcal{F}^\vee}^{X \times X \rightarrow X}(L \boxtimes L)) = h^i(X \times X \times X, L^2 \boxtimes L \boxtimes L^{-1}) = \begin{cases} h^0(X, L)^2 h^0(X, L^2) & \text{for } i = \dim X \\ 0 & \text{otherwise} \end{cases}$$

Since the degree of the isogeny φ_L is $h^0(X, L)^2$, we get (5.4).

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